CHAPTER

1 Basic Mathematics and Logarithm

Some Important Identities

1. $(a+b)^2 = a^2 + 2ab + b^2 = (a-b)^2 + 4ab$
2. $(a-b)^2 = a^2 - 2ab + b^2 = (a+b)^2 - 4ab$
3. $a^2 - b^2 = (a+b)(a-b)$
4. $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
5. $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$
6. $a^3 + b^3 = (a+b)^3 - 3ab(a+b) = (a+b)(a^2+b^2-ab)$
7. $a^3 - b^3 = (a-b)^3 + 3ab(a-b) = (a-b)(a^2+b^2+ab)$
8. $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca = a^2+b^2+c^2+ 2abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$
9. $a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$
10. $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$
    $= \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$
    If $a+b+c=0$, then $a^3+b^3+c^3=3abc$
11. $a^4 - b^4 = (a+b)(a-b)(a^2+b^2)$
12. $a^4 + a^2 + 1 = (a^2+1)^2 - a^2 = (1+a+a^2)(1-a+a^2)$

Laws of Indices

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$
3. $(a^m)^n = (a^n)^m = a^{mn}$
4. $\left(\frac{a}{b}\right)^{\frac{m}{n}} = \left(\frac{b}{a}\right)^{-\frac{m}{n}}$
5. $a^m \div b^{-n} = a^m \times b^n$
6. $(\sqrt[n]{a})^n = a$, where, $n \in N$, $n \geq 2$ and a is positive rational number
7. $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$, where $n \in N$, $n \geq 2$ and a, b are rational number
8. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, $a, b \in R$ and atleast one of a or b should be positive.

Ratio and Proportion

a, b, c, d are in proportion. Then,

(i) $\frac{a}{b} = \frac{c}{d}$

(ii) $\frac{a}{c} = \frac{b}{d}$ ....(alternendo)

(iii) $\frac{a+b}{b} = \frac{c+d}{d}$ ....(componendo)

(iv) $\frac{a-b}{b} = \frac{c-d}{d}$ ......(dividendo)

(v) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ ......(componendo and dividendo)

(vi) $\frac{b}{a} = \frac{d}{c}$ ....(invertendo)

(vii) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} =$ ......, then each

$$\frac{a+c+e+......}{b+d+f+......} = \frac{\text{Sum of the numerators}}{\text{Sum of the denominators}}$$

Solving of Inequalities by Wavy Curve Method

Step 1: Obtain critical points by equating all factors to zero.

Step 2: Plot the critical points on the number line in the increasing order.

Step 3: Put plus sign in the right most interval.

Step 4: Now, if a root is repeated even times, the sign of the function will remain the same in the two adjacent sub-intervals of the root. (when we are moving from right to left)

Step 5: If a root is repeated for odd times the sign of the function will be different in the two adjacent sub intervals of the root. (when we are moving from right to left)

Properties of Logarithm

1. $\log_e(ab) = \log_e a + \log_e b$; $(a, b > 0)$
2. $\log_e\left(\frac{a}{b}\right) = \log_e a - \log_e b$; $(a, b > 0)$
3. $\log_e a^m = m\log_e a$; $(a > 0, m \in R)$
4. $\log_a a = 1$; $(a > 0, a \neq 1)$
5. $\log_{b^m} a = \frac{1}{m}\log_b a$; $(a, b > 0, b \neq 1$ and $m \in R - \{0\})$
6. $\log_b a = \frac{1}{\log_a b}$; $(a, b > 0$ and $a, b \neq 1)$

7. $\log_b a = \dfrac{\log_m a}{\log_m b}$; ($a, b, m > 0$ and $m, b \neq 1$)

8. $a^{\log_a m} = m$; ($a > 0, a \neq 1$ $m > 0$)

9. $a^{\log_c b} = b^{\log_c a}$; ($a, b, c > 0$ and $c \neq 1$)

10. If $\log_m x > \log_m y \Rightarrow \begin{cases} x > y, if \;\; m > 1 \\ x < y, if \; 0 < m < 1 \end{cases}$

($m, x, y > 0$, $m \neq 1$)

11. $\log_m a = b \Rightarrow a = m^b$; ($m, a > 0$, $m \neq 1 \in$ real number)

12. $\log_m a > b \Rightarrow \begin{cases} a > m^b ; if \;\; m > 1 \\ a < m^b ; if \;\; 0 < m < 1 \end{cases}$

13. $\log_m a < b \Rightarrow \begin{cases} a < m^b ; if \;\; m > 1 \\ a > m^b ; if \;\; 0 < m < 1 \end{cases}$

Logarithmic Equations

(i) An equation of the form $\log_a f(x) = b$, ($a > 0$), $a \neq 1$ is equivalent to the equation. $f(x) = a^b$, ($f(x) > 0$)

(ii) If $\log_a f(x) > \log_a g(x)$ and $a > 1$, then $\Rightarrow \begin{cases} g(x) > 0 \\ f(x) > 0 \\ f(x) > g(x) \end{cases}$

(iii) If $\log_a f(x) > \log_a g(x)$ and $a < 1$, then $\Rightarrow \begin{cases} f(x) > 0 \\ g(x) > 0 \\ f(x) < g(x) \end{cases}$

Definition of Modulus

$$|x| = \begin{cases} x, \; if \;\; x \geq 0 \\ -x, \;\; if \; x < 0 \end{cases}$$

Properties of Modulus

Let 'a', 'b' are positive real number then,

(i) $|x| = a \Rightarrow x = \pm a$

(ii) $|x| \leq a \Rightarrow -a \leq x \leq a \Rightarrow x \in [-a, a]$

(iii) $|x| < a \Rightarrow -a < x < a \Rightarrow x \in (-a, a)$

(iv) $|x| \geq a \Rightarrow x \leq -a$ or $x \geq a \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

(v) $|x| > a \Rightarrow x < -a$ or $x > a \Rightarrow x \in (-\infty, -a) \cup (a, \infty)$

(vi) $a \leq |x| \leq b \Rightarrow x \in [-b, -a] \cup [a, b]$

(vii) $a < |x| < b \Rightarrow x \in (-b, -a) \cup (a, b)$

($viii$) $|x + y| \leq |x| + |y|$; equality holds when $xy \geq 0$

(ix) $|x - y| \geq ||x| - |y||$; equality holds when $xy \geq 0$

(x) $|x + y| \geq ||x| - |y||$; equality holds when $xy \leq 0$

(xi) $|x - y| \leq |x| + |y|$; equality holds when $xy \geq 0$